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NOTE ON AN ANALYTICAL CRITERION FOR ELASTIC NORMAL MODE OSCILLATION†

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Abstract—A definition of a stationary mode elastic oscillation of structure is derived based on a variational criterion which in turn is derived from the uniqueness proof for such problems. For materials in which stress is a homogeneous function of strain, e.g. a power-lawed relation in the uniaxial form, the stationary modes coincide with the normal modes. This variational approach provides some physical insight into the characteristics of an elastic oscillation, e.g. Rayleigh's minimum theorem, and is entirely consistent with the concept of approximation based on a truncated series in which only a certain number of preferred modes are taken.

1. INTRODUCTION

THE vectorial (Newtonian) and the variational (Euler–Lagrangian) theories of mechanics are two different mathematical descriptions of the same realm of natural phenomena [1]. Some important specific behaviours of a certain system, e.g. the elastic normal mode vibration, may also be described from the different point of view of these two theories. The vectorial analysis of the normal mode vibration may be investigated by some mathematical techniques, such as the separation of variables. In this note an attempt is made to explore a variational criterion for the normal mode oscillation.

The elastic stress–strain relation is assumed to be:

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}} \tag{1}$$

where W is the strain energy density. For the case of linear elasticity, equation (1) reduces to

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

where C_{ijkl} is a positive definite symmetric tensor.

Consider an elastic oscillation problem, i.e. initial displacements u_i^0 and initial velocities $\dot{u}_i^0$ are prescribed and therefore no external forces do work on the structure. Consider the solutions to two problems for identical structures, differing only in the initial conditions, u_i^{a0}, $\dot{u}_i^{a0}$ and u_i^{b0}, $\dot{u}_i^{b0}$. By the principle of virtual velocities, where m is the mass density, then

$$-\int_V m(\ddot{u}_i^a - \ddot{u}_i^b)(\dot{u}_i^a - \dot{u}_i^b)\,\mathrm{d}V = \int_V (\sigma_{ij}^a - \sigma_{ij}^b)(\dot{\varepsilon}_{ij}^a - \dot{\varepsilon}_{ij}^b)\,\mathrm{d}V \tag{3}$$

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Consider first the linear elastic case, i.e. equation (2), (3) can be written as

$$-\frac{\mathrm{d}\Delta}{\mathrm{d}t} = 0 \tag{4}$$

where

$$\Delta = \Delta_K + \Delta_V$$

$$\Delta_K = \int_V \frac{m}{2}(\dot{u}_i^a - \dot{u}_i^b)(\dot{u}_i^a - \dot{u}_i^b)\,\mathrm{d}V \geq 0 \tag{5a}$$

$$\Delta_V = \int_V \frac{1}{2}(\sigma_{ij}^a - \sigma_{ij}^b)(\varepsilon_{ij}^a - \varepsilon_{ij}^b)\,\mathrm{d}V \geq 0 \tag{5b}$$

Δ may be considered as a measure of difference between the two solutions. Since the system is linear, the difference between two solutions $u_i(X, t) = u_i^a - u_i^b$, $\dot{u}_i(X, t) = \dot{u}_i^a - \dot{u}_i^b$ is also a solution of the problem. In terms of u_i and $\dot{u}_i$, Δ and $\mathrm{d}\Delta/\mathrm{d}t$ become

$$\Delta = \Delta_K + \Delta_V = K + V = E \tag{6a}$$

$$-\frac{\mathrm{d}\Delta}{\mathrm{d}t} = -\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mathrm{d}K}{\mathrm{d}t} - \frac{\mathrm{d}V}{\mathrm{d}t} = 0 \tag{6b}$$

and

$$\Delta_U = \int_V \frac{m}{2}(u_i^a - u_i^b)(u_i^a - u_i^b)\,\mathrm{d}V = U \tag{6c}$$

where K, V, E and U are respectively the kinetic energy, the strain energy, the total energy and the total amount of the square of displacement, of the solution u_i, $\dot{u}_i$. Uniqueness of solution will follow then from observing that the initial conditions are the same for both solutions, i.e. $u_i^{a0} = u_i^{b0}$ and $\dot{u}_i^{a0} = \dot{u}_i^{b0}$ throughout the body. This implies that $\Delta \equiv 0$ and in turn that the two solutions are identical. If, however, the two solutions do not have the same initial conditions, but that they do satisfy the same boundary conditions, then one solution may be considered as an approximation for the other with a value of difference or error Δ.

It is clear that the best approximate solution may be the one which makes Δ minimum. However, if an attempt is made to formulate the problem in as general terms as possible, one is led to the undeniable but impractical conclusion that the best approximation is the actual solution of the problem, i.e. $u_i = u_i^a - u_i^b = 0$, $\dot{u}_i = \dot{u}_i^a - \dot{u}_i^b = 0$ and hence $\Delta = 0$. In order to ensure that the approximate solution is not chosen from a set of fields which contains the actual solution, it is reasonable to compare all the possible fields based on certain criterion. A reasonable constraint will be that $\Delta = \text{const}$. Our next problem will be then to find a reasonable measure of difference between two solutions.

At any state of oscillation, the displacement and the velocity fields are the basic independent variables of the system. Hence, the displacement and the velocity fields may be treated separately. A measure of difference between two "static deformations", $u_i^a(X, t)$ and $u_i^b(X\ t)$, is defined by $\Delta_U = U$ (approximation by least square method). Similarly, the measure of difference between two "kinematic deformations", $\dot{u}_i^a(X, t)$ and $\dot{u}_i^b(X, t)$, is defined by $\Delta_K = K$.

Consider first the displacement field. It would be reasonable to judge the goodness of an approximate displacement field by the smallness of Δ_U based on an isometric constraint Δ_V = const. This constraint condition follows from the condition Δ = const. Then the most important displacement field u_i which should not be omitted from the approximation will be the one which maximizes $U = \Delta_U$ subject to the condition that $V = \Delta_V$ is held constant. Equivalently, in order to have the same goodness of approximation, i.e. $U = \Delta_U$ = const., the most important displacement field u_i will be the one which minimizes the strain energy $V = \Delta_V$.

Consider next the velocity fields. If $-\mathrm{d}\Delta_K/\mathrm{d}t$ is large, then the difference between the two velocity fields will decrease rapidly. In consequence, the most important velocity field which should not be omitted from the approximation will be the one which makes $K = \Delta_K$ maximum or $-\mathrm{d}K/\mathrm{d}t = -\mathrm{d}\Delta_K/\mathrm{d}t$ minimum. In order to avoid the similar impractical conclusion, the most important velocity field $\dot{u}_i$ should be the one which minimizes $-\mathrm{d}K/\mathrm{d}t = -\mathrm{d}\Delta_K/\mathrm{d}t$ subject to the condition that $K = \Delta_K$ is held constant.

The arguments given above are carried out for the linear elastic case. However, for non-linear materials, it is hypothetized that Δ behaves almost in the same way as E and $\mathrm{d}\Delta/\mathrm{d}t$ behaves almost in the same way as $\mathrm{d}E/\mathrm{d}t$. Then the criteria are defined in the same way for both linear and non-linear materials.

The combined criteria given above for displacement and velocity fields lead to a variational problem whose solutions referred to as *stationary mode solutions* will be discussed in the following section. It will be shown that for some forms of constitutive relation, the stationary mode solutions coincide with that of a normal mode oscillation.

The application of these concepts of stationary mode for approximating solution of elastic oscillation problem will be studied. The ideas of the approximation are based on the same concepts which are discussed in detail in [2] and [3].

2. STATIONARY MODE SOLUTIONS

The stationary mode is defined as the kinematically admissible displacement field in which the strain energy is less than that in all neighboring kinematically admissible displacement fields which possess the same amount of U, i.e.

$$J_1 = \int_V W(\varepsilon_{ij})\,\mathrm{d}V - \lambda_1\left[\int_V \frac{m}{2}u_i u_i\,\mathrm{d}V - U\right] = \text{minimum} \tag{7a}$$

where λ_1 is the Lagrangian multiplier and it is assumed that $\partial^2 W/\partial\varepsilon_{ij}\partial\varepsilon_{kl}$ is a positive definitive symmetric tensor and hence the extremal values of V are relative minimal. As to the velocity field, the stationary mode solution is defined as the kinematically admissible velocity field in which the rate of decrease of the kinetic energy (or the rate of change of the strain energy) is minimum in comparison with that in all neighboring kinematically admissible velocity fields which possess the same total kinetic energy, i.e.

$$J_2 = \int_V \frac{\partial W}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij}\,\mathrm{d}V - \lambda_2\left[\int_V \frac{m}{2}\dot{u}_i\dot{u}_i\,\mathrm{d}V - K\right] = \text{minimum.} \tag{7b}$$

The Euler equations of (7), after using equation (1), are

$$\frac{\partial\sigma_{ij}}{\partial X_j} + \lambda_1 m u_i = 0, \tag{8}$$

and

$$\frac{\partial \sigma_{ij}}{\partial X_j} + \lambda_2 m \dot{u}_i = 0 \tag{9}$$

since σ_{ij} is independent of the variation of velocity. From equations (8) and (9), the following relations can be obtained:

$$\dot{u}_i = \frac{\lambda_1}{\lambda_2} u_i \tag{10}$$

$$K = \left(\frac{\lambda_1}{\lambda_2}\right)^2 U \tag{11}$$

$$\int_V (W+\Omega)\,\mathrm{d}V = \int_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = 2\lambda_1 U \tag{12}$$

and

$$-\frac{\partial K}{\mathrm{d}t} = \int_V \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}V = 2\lambda_2 K \tag{13}$$

where Ω is the complementary energy density.

Equation (7) or equations (8) and (9), thus, provide a displacement field $u_i(X, U)$ for a specific level of U and also a velocity field $\dot{u}_i(X, K)$ for a specific level of K. These two criteria are further connected by the conservation of energy,

$$K + V = E = \text{total energy} = \text{const.} \tag{14}$$

Hence, if the initial conditions are given, equations (11), (13) and (14) permit solutions for $K(t)$, $V(t)$ and $U(t)$, and then the stationary mode solutions $u_i(X, t)$ and $\dot{u}_i(X, t)$.

Equation (13) can be regarded as a necessary condition for equilibrium since it can be obtained from equation (9) and the equation of motion $\partial\sigma_{ij}/\partial X_j = m\ddot{u}_i$ (body forces are neglected). It is not, however, a sufficient condition, and the stationary mode solution may not be dynamically admissible. It will be shown that if the material is such that σ_{ij} is a homogeneous function of ε_{ij} of order n, say, then equation (13) is also a sufficient condition for equilibrium since the Euler equations can be reduced to a local equilibrium equation as follows:

If σ_{ij} is a homogeneous function of ε_{ij} of order n, then the Euler solution of (8) can be expressed as

$$u_i(X, U) = \gamma(U)\phi_i(X). \tag{15}$$

Substituting (15) into (11) and differentiating both sides with respect to K, it gives

$$\frac{\partial}{\partial K}\left(\frac{\lambda_1 \gamma}{\lambda_2}\right) = \frac{\gamma}{2\sqrt{(KU)}}. \tag{16}$$

Using equations (13), (16), (11) and (8), the acceleration of the stationary mode solution is, from equation (10),

$$\frac{\partial \dot{u}_i}{\partial t} = \frac{\partial}{\partial K}\left(\frac{\lambda_1 \gamma}{\lambda_2}\right)\frac{\partial K}{\partial t}\phi_i(X) = -\lambda_1 u_i = \frac{1}{m}\frac{\partial \sigma_{ij}}{\partial X_j}. \tag{17}$$

Hence, the stationary mode solution is both kinematically and dynamically admissible, and therefore, the stationary mode solution describes the behaviour of a single normal mode oscillation, and it has the following relations:

$$\Omega = nW \tag{18}$$

$$V = \frac{2}{1+n}\lambda_1 U, \qquad \lambda_1 = \lambda_1(U) \tag{19}$$

$$\lambda_2(U) = \frac{\lambda_1}{\sqrt{[(E/U)-(2\lambda_1/1+n)]}} \tag{20}$$

For the linear elastic case, i.e. $n = 1$, equation (19) becomes

$$\lambda_1 = \frac{\int_V \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\,\mathrm{d}V}{\int_V (m/2)u_i u_i\,\mathrm{d}V} = \frac{\int_V C_{ijkl}\phi_{i,j}\phi_{kl}\,\mathrm{d}V}{\int_V m\phi_i\phi_i\,\mathrm{d}V} \tag{21}$$

Hence, λ_1, the square of the natural frequency, is stationary (relative minimal) about the normal mode and is independent of the amplitude. Equation (21) gives then Rayleigh's minimum theorem (e.g. [4]).

3. APPLICATION FOR APPROXIMATE SOLUTIONS

As discussed in Section 1, the stationary mode distributions are the most important fields which should not be omitted from the approximation since they have more capacity for storing [see equation (7a)] and preserving [equation (7b)] energy in a body. Therefore, they may be used legitimately as an approximate solution. In the case of linear elasticity where modes may be superposed, the solution is approximated by a truncated series, and it was shown in [2] that the mode approximations are entirely consistent with the established view of normal mode analysis.

Suppose that the stationary mode solution is now used to approximate an unknown solution u_i^*, $\dot{u}_i^*$. Since u_i^* and $\dot{u}_i^*$ are not known for $t > 0$, the initial conditions provide the only possible information. In such cases, the initial amplitudes γ_0 and $\dot{\gamma}_0$ of the approximate mode solution $u_i(X, t) = \gamma(t)\phi_i(X)$, $\dot{u}_i = \dot{\gamma}\phi_i(X)$ can be regarded as parameters which may be varied in order to provide the best approximation, i.e. to make the initial differences

$$\Delta_U^0 = \int_V \frac{m}{2}(u_i^{*0}-u_i^0)(u_i^{*0}-u_i^0)\,\mathrm{d}V, \tag{22a}$$

and

$$\Delta_K^0 = \int_V \frac{m}{2}(\dot{u}_i^{*0}-\dot{u}_i^0)(\dot{u}_i^{*0}-\dot{u}_i^0)\,\mathrm{d}V \tag{22b}$$

as small as possible. Hence, γ_0 and $\dot{\gamma}_0$ are determined by

$$\frac{\mathrm{d}\Delta_U^0}{\mathrm{d}\gamma_0} = 0 \Rightarrow \gamma_0 = \frac{\int_V m u_i^{*0}\phi_i\,\mathrm{d}V}{\int_V m\phi_i\phi_i\,\mathrm{d}V} \tag{23a}$$

$$\frac{\mathrm{d}\Delta_K^0}{\mathrm{d}\dot{\gamma}_0} = 0 \Rightarrow \dot{\gamma}_0 = \frac{\int_V m\dot{u}_i^{*0}\phi_i\,\mathrm{d}V}{\int_V m\phi_i\phi_i\,\mathrm{d}V} \tag{23b}$$

These imply that

$$\int_V m(u_i^{*0} - u_i^0)\phi_i \, dV = 0 = \int_V m(\dot{u}_i^{*0} - \dot{u}_i^0)\phi_i \, dV \tag{24}$$

$$\Delta_U^0 = U^{*0} - U^0 \geq 0, \qquad \Delta_K^0 = K^{*0} - K^0 \geq 0 \tag{25}$$

Equation (24) shows that the mode shape $\phi_i(X)$ is orthogonal to the differences, $u_i^{*0} - u_i^0$ and $\dot{u}_i^{*0} - \dot{u}_i^0$ between these two initial fields. Also, equation (25) shows that the average of the differences is equal to the difference between the averages.

One further useful piece of information can be obtained in the form of upper bound on the period of oscillation. Suppose that the initial conditions are $u_i^{*0} = 0$ and $\dot{u}_i^{*0}$ prescribed throughout the body. From (7b), at any level of kinetic energy, the actual rate of decrease of kinetic energy is not less than that of the primary stationary mode solution. This implies that

$$T^* \leq T \tag{26}$$

where T^* is the period of oscillation of the actual solution and T is the period of the primary stationary mode oscillation in which the initial conditions are found by putting $K^0 = K^{*0}$ rather than by minimizing Δ_K^0.

To illustrate the application of the stationary mode for approximate solutions, problems of uniform beam subjected to transverse impulse as shown in Figs. 1(a) and 1(b) will be given. The moment–curvature relation, $M \sim \kappa$, will be assumed to be a power law type as, Fig. 1(c),

$$\frac{\kappa}{\kappa_0} = \left(\frac{M}{M_0}\right)^P, \qquad \kappa = -\frac{\partial^2 u}{\partial^2 X} \tag{27}$$

where κ_0 and M_0 are constants and P will be taken to be an odd positive integer. It can be easily shown that relation (27) ensures the existence of a normal mode oscillation. In the following analysis, the elementary beam theory will be used.

Consider first the simply supported beam subjected to a uniform impulse $I = mV_0$, Fig. 1(a). The mode shape $\phi(X)$ may be found by applying equation (7a) or directly by seeking solutions to the Euler equation (8), i.e. $u = \gamma\phi(X)$ and

$$\frac{d^2}{d^2\xi}\left[\left(-\frac{d^2\phi}{d^2\xi}\right)^{1/P}\right] = -A\phi \tag{28}$$

where $\xi = X/l$, m = mass per unit length, and

$$A = \frac{ml^2(\kappa_0 l^2)^{1/P}}{M_0}\lambda_1\gamma^{1-1/P} = \text{const.} \tag{29}$$

No closed form solution can be found to this equation for the nonlinear case and a simple numerical procedure was used to solve for $\phi(X)$. The following procedure appears to converge very rapidly:

(i) guess a function $\phi^0(X)$ for the right hand side of (28);
(ii) integrate twice, taking account of boundary conditions on $d^2\phi/d\xi^2$;
(iii) raise this function to the power P;
(iv) integrate twice, taking into account the boundary conditions on ϕ;
(v) normalize the resulting function, i.e. $\phi(l/2) = 1$, giving ϕ^1 and the normalization factor A.

ϕ^1 becomes a new guessed function, and the process is repeated.

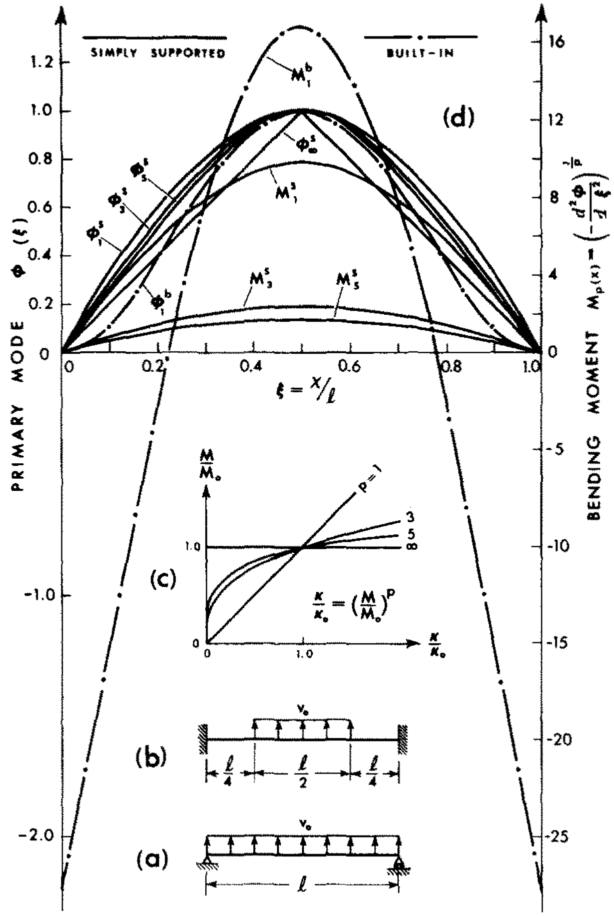

FIG. 1.

A good approximation (after the first cycle in the above procedure) to the primary modes for the cases $P = 3$ and $P = 5$ may be given analytically:

$$P = 1, \qquad \phi_1^S(\xi) = \sin \pi\xi, \qquad A_1 = \pi^4$$

$$P = 3, \qquad \phi_3^S(\xi) = \frac{1}{1{\cdot}037}(\sin \pi\xi - \tfrac{1}{27}\sin 3\pi\xi), \qquad A_3 = 23{\cdot}56 \tag{30}$$

$$P = 5, \qquad \phi_5^S(\xi) = \frac{1}{1{\cdot}060}(\sin \pi\xi - \tfrac{1}{18}\sin 3\pi\xi + \tfrac{1}{250}\sin 5\pi\xi), \qquad A_5 = 17{\cdot}91$$

The primary modes and their corresponding moment distributions are shown in Fig. 1(d). From equations (10), (15) and (20), it can be shown that the normal mode solution $\gamma(t)$ is obtained by the ordinary differential equation

$$\frac{\dot{\gamma}}{V_0} = \sqrt{\left[\left(\frac{\gamma_0}{V_0}\right)^2 - \frac{2P}{1+P}\frac{A}{\eta}\left(\frac{\gamma}{\kappa_0 l^2}\right)^{1+1/P}\right]} \tag{31}$$

where

$$\eta = \frac{mV_0^2}{M_0\kappa_0}$$

is the dynamic parameter indicating the intensity of the initial impulse and γ_0 is the "best" value of the initial amplitude of the approximate normal mode solution [see equation (23a)]. Equation (31) was solved by a Runge–Kutta procedure (e.g. [5]), and the normal mode solutions are plotted in Figs. 2–4.

The equation of motion of the actual dynamic problem, in a non-dimensional form, is

$$\frac{\partial^2}{\partial\xi^2}\left[-\frac{\partial^2}{\partial\xi^2}\left(\frac{u}{\kappa_0 l^2}\right)\right]^{1/P} = \eta\frac{\partial^2}{\partial\tau^2}\left(\frac{u}{\kappa_0 l^2}\right) \tag{32}$$

where the dimensionless time $\tau = \sqrt{(M_0/ml^4\kappa_0)}t$. The actual dynamic solutions were carried out by the finite-difference method and a Runge–Kutta procedure and were plotted in Figs. 2–4. Figures 2–4 show the comparisons of the central displacements and the central velocities between the actual solution and the approximate normal mode solution.

In this example, the linear case, $P = 1$, has the smallest value of Δ/K^{*0} and this value increases as the behaviour of the material becomes more highly nonlinear. This point is expected since the primary mode of the linear case is closest to the uniform distribution of the initial impulse.

Consider next the built-in supported beam subjected to impulsive loading over a length $l/2$ symmetric about the midpoint, so that the initial velocity is V_0 on this section and zero

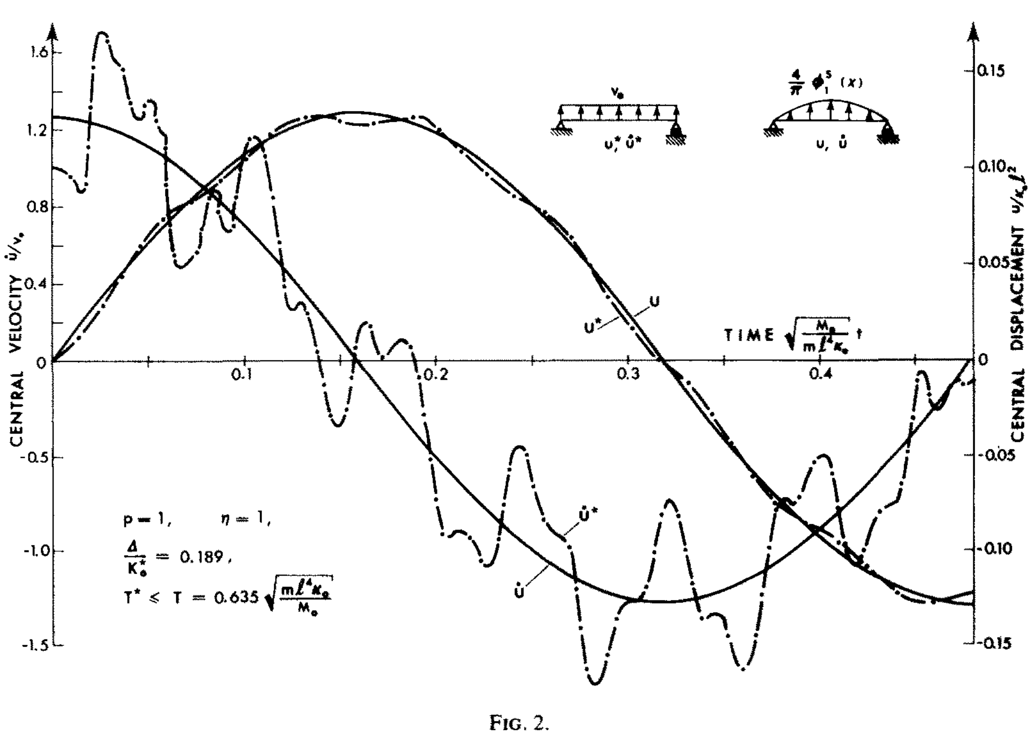

FIG. 2.

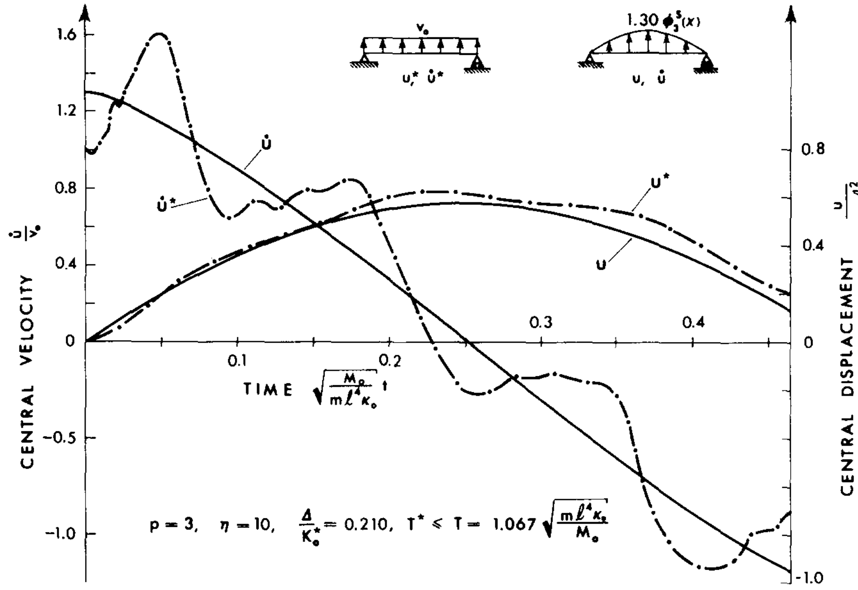

FIG. 3.

in the remaining length $l/2$ as shown in Fig. 1(b). In this example, only the linear case, $P = 1$, will be considered. The calculating procedures are exactly the same as that in the previous example except the boundary conditions. In solving the mode shape $\phi_1^b(X)$, no boundary conditions on $d^2\phi/d\xi^2$ in step (ii) are available, while there are four needed boundary conditions on ϕ and $d\phi/d\xi$ in step (iv).

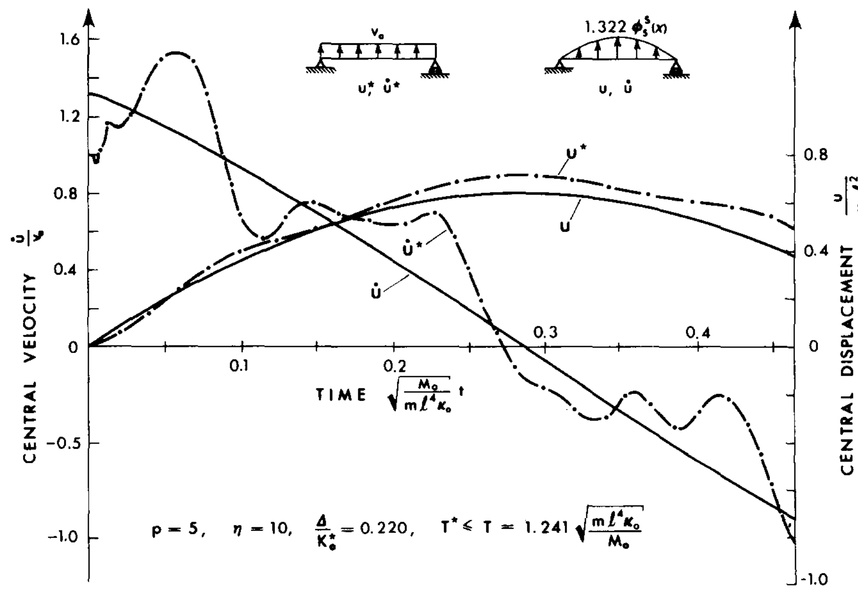

FIG. 4.

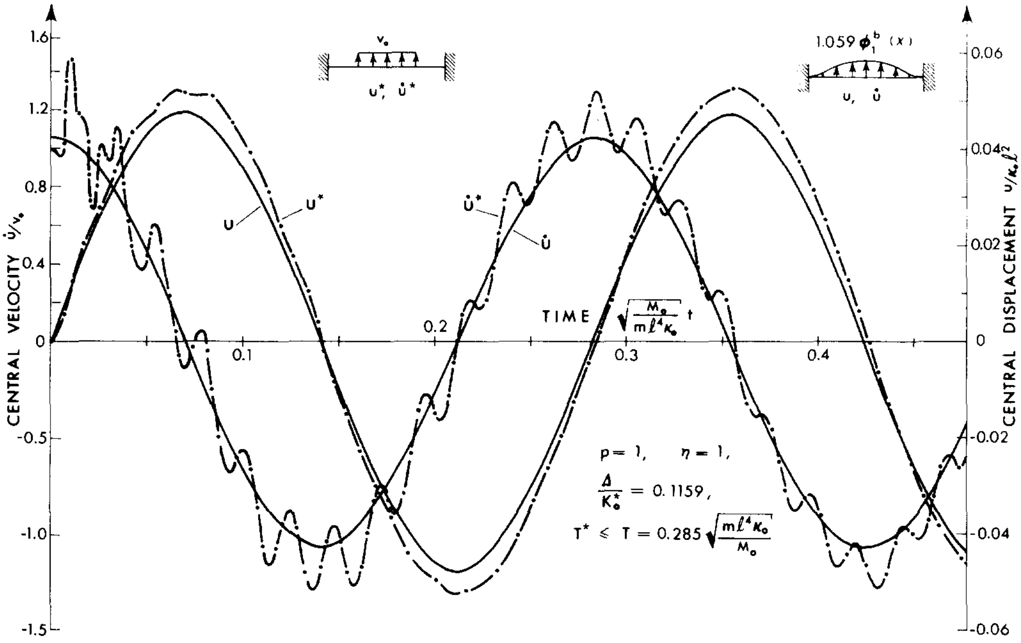
CENTRAL VELOCITY $\dot{u}/v_o$
CENTRAL DISPLACEMENT $u/\kappa_o\ell^2$
TIME $\sqrt{\frac{M_o}{m\ell^4\kappa_o}}\,t$
v_o
u^*, $\dot{u}^*$
$1.059\,\phi_1^b(x)$
u, $\dot{u}$
$p = 1$, $\eta = 1$,
$\frac{\Delta}{K_o^*} = 0.1159$,
$T^* \leq T = 0.285\sqrt{\frac{m\ell^4\kappa_o}{M_o}}$

FIG. 5.

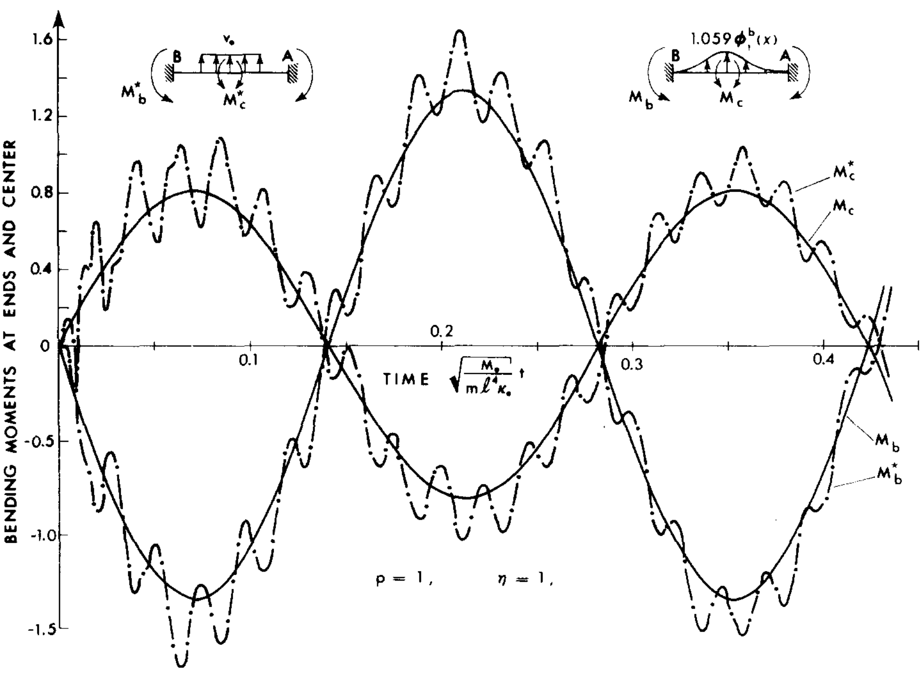
BENDING MOMENTS AT ENDS AND CENTER
TIME $\sqrt{\frac{M_o}{m\ell^4\kappa_o}}\,t$
v_o
B A
M_b^* M_c^*
$1.059\,\phi_1^b(x)$
M_b M_c
$p = 1$, $\eta = 1$,

FIG. 6.

For the build-in problem, $A_1 = 492{\cdot}9$ for $P = 1$. The primary mode and the moment distribution $M_1^b(X)$ are shown in Fig. 1(d). The central displacement–time and velocity–time histories for $\eta = 1$ are shown in Fig. 5. Figure 6 shows the comparisons of the bending moments at the ends and the centre between the actual and the approximate solutions. Figure 7 shows the bending moment distributions of these two solutions at time $\tau = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$ and $0{\cdot}4$. It can be seen that the inflection point remains at $\xi = 0{\cdot}226$ almost all the time.

From the examples, it is seen that, in general, the stationary mode solution may provide not only a good approximation in an average sense to an impulsive loading problem but also some physical insight into the features of the problem.

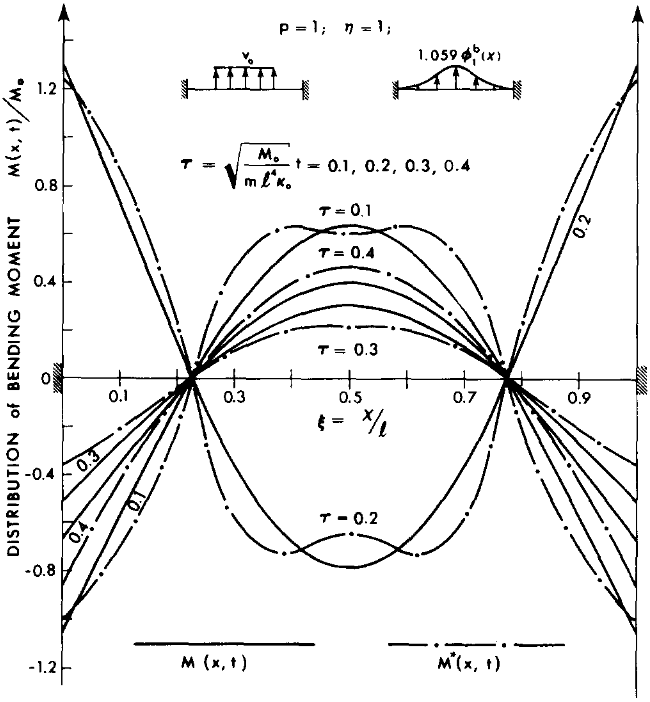

FIG. 7.

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Абстракт—Выводится определение стационарного вида упругого колебания конструкции, на основе вариационного критерия, которое последственно определенное из доказательства единственности для такихже задач. Для материалов, в которых нрпряжение являетcя однородной функцией деформации (например степенная зависимость в одноcной форме), стационарные виды колебании согласуются с нормальными видами колебаний. Этот вариационный подход дает некоторую физическую способность точного исследования характеристик упругого колебания, например теорему минимума Релея. Этот подход также целиком согласовывается с концепцией аппроксимации, основанной на отбрасывании членов в рядах, которых учитывается некоторое число необходимых видов колебаний.